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THE ROYAL SOCIETY

Behaviour of liquefied sand

By D. Kolymbas

Institute for Geotechnics and Tunnelling, University of Innsbruck, Techniker Strasse 13, A-6020 Innsbruck, Austria

Liquefied sand is a fluid rather than a solid. This fluid, however, has a limited lifetime since a process of solidification (also called consolidation) sets in immediately after liquefaction. Consolidation means a partial segregation of pore fluid and grain skeleton. Sand volcanoes, often observed after liquefaction, are interpreted as bifurcated modes of consolidation. The role of liquefaction in landslides and debris flows is analysed, and an additional mode of 'dry' liquefaction is introduced with the aim of explaining debris flows that occur in the absence of pore fluid (e.g. on the Moon).

Keywords: liquefaction; fluidization; sand; avalanches; sand boils; contractancy

1. Introduction

Liquefied sand is a peculiar medium occupying a transitional state between solid and liquid. A considerable amount of practical and theoretical work in soil mechanics has been devoted to describing and predicting the onset of liquefaction (also called 'fluidization' in chemical engineering†), which is understood as a loss of stability of loose water-saturated sand deposits and other granular media mainly induced by dynamic impacts (e.g. earthquakes (Bardet & Kapuskar 1993; Kuribayashi & Tatsuoka 1977)). Liquefaction is now recognized as one of the main sources of earthquake-induced catastrophes (see figures 1 and 2).

Liquefaction can also be released by vibrations and other activities at construction sites (see, for example, Heil & Möller 1992; Müller-Kirchenbauer *et al*. 1992). Although the conditions under which liquefaction can occur are now well understood, or, at least, recognizable, we know very little about the behaviour of liquefied soil, e.g. what happens *after* the onset of liquefaction. Generally, it is expected that liquefied soil will subsequently compact. It turns out that this compaction proceeds along with very peculiar mechanisms which can be considered as particular cases of pattern formation. One of these mechanisms is the eruption of pore fluid from localized outlets, which are considered as sand volcanos or sand boils (see figures 3 and 4). Kuribayashi & Tatsuoka (1977) report:

> … sand boils, sand volcanos and floating up of wooden piles and caissons during the earthquake Nr. 15 were reported …
>
> The most violent water spouting during the earthquake Nr. 4 was observed at the site … where water with sand ejected high above 2 m from wells and sands deposited over the roofs of nearby houses.

† The distinction between the synonyms 'liquefaction' and 'fluidization' encountered in some papers (e.g. Lowe 1975) is not reasonable.

Figure 1. Liquefaction-induced damage following an earthquake in Caracas.

Figure 2. Liquefaction disaster.

The recent discovery by McKenzie (1984; see also Scott & Stevenson 1984; Scott *et al*. 1986) that real volcanos are also the result of the segregation of liquid magma from a granular rock matrix, reveals that the considered effect is not limited to saturated soils but also occurs in other geometries and time-scales with completely different mixtures.

Besides the eruptive segregation of pore fluid from the granular matrix, and the instabilities of fluidized beds† (see, for example, Garg & Pritchett 1975; Jackson 1985; Nichols *et al*. 1994; Pritchett *et al*. 1978), the peculiarity of liquefied soil is manifested through mudflows, turbidity currents and similar phenomena of rapid movements of large water-saturated granular masses.

† For a water-saturated granular medium it can be shown by perturbation analysis that initial disturbances grow exponentially with time if the stiffness of the grain skeleton vanishes.

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Figure 3. Eruption of pore water after an underground explosion.

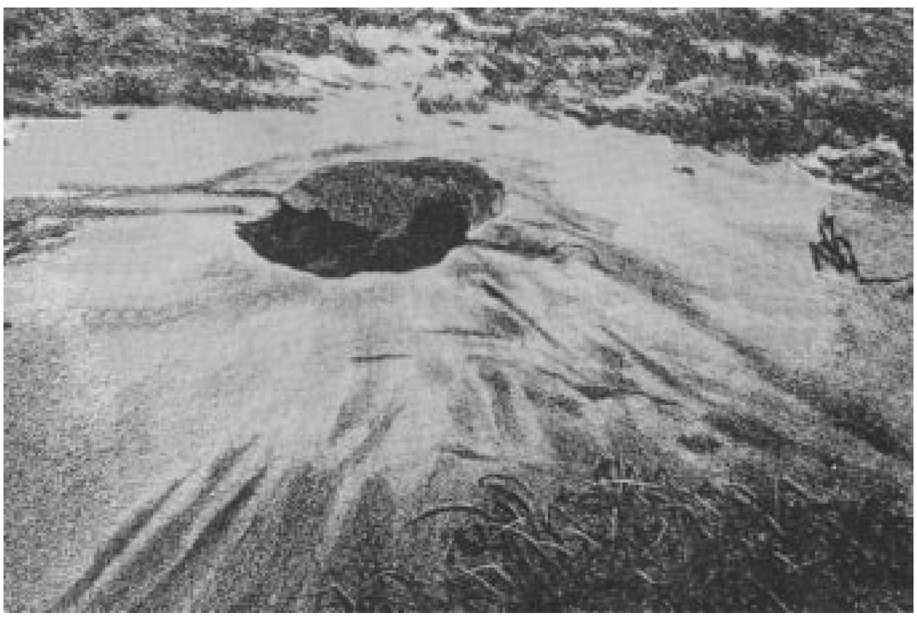

Figure 4. Sand crater, formed by the eruption of a sand volcano.

2. Consolidation theories

The contractant deformation of fluid-saturated porous media is described by the so-called theory of consolidation. According to their basic assumptions, there are several consolidation theories. Most of them assume that the granular matrix is elastic. Despite their high level of mathematical intricacy, they are so far incapable of analysing phenomena occurring in liquefied soil. These phenomena are partly of a dynamic nature (such that acceleration terms are indispensable), and can be considered as pattern formations related to the non-uniqueness of the pertinent initial boundary-value problem (cf. the erosion structure as shown in figure 5).

In most cases, consolidation is considered with only one spatial direction, z, and, therefore, any exceptional water-escape structures are *a priori* excluded. Terzaghi's (1943) derivation of the governing equation (which is identical to the heat equation) starts from the mass-balance equation of water,

$$\frac{1}{(1+e)}\frac{\partial e}{\partial t} + \frac{\partial v}{\partial z} = 0,$$

where e is the void ratio and v is the filter velocity of water. With Darcy's law

$$v = -k\frac{\partial h}{\partial z}$$

and with $k = \text{const.}$,

$$\sigma = \sigma' + u = \text{const.}, \qquad \mathrm{d}h = \mathrm{d}u/\gamma_{\mathrm{w}}, \qquad \mathrm{d}e = a_v \mathrm{d}\sigma',$$

we obtain $u_t = c_v u_{zz}$, with $c_v := k(1+e)/(a_v\gamma_{\mathrm{w}}) = \text{const.}$ and $a_v = \text{const.}$ expressing the compressibility of the granular matrix. σ is the total stress, σ' is the so-called effective stress (defined by $\sigma' = \sigma - u$), and u is the pore pressure. σ, σ' and u are here assumed positive for compression.

Terzaghi himself generalized his original consolidation equation, $u_t = c_v u_{zz}$, for three spatial dimensions into $u_t = c_v \nabla^2 u$ (Terzaghi 1943). This generalization, however, is incorrect (or incomplete) for general boundary conditions for the following reason (see also Viggiani 1967; Verruijt 1995, pp. 29–30); the coupling between the pore pressure, u, appearing in Darcy's equation and the effective stress, σ', which governs the deformation of the granular matrix, is achieved by setting the total stress, $\sigma_{ij} = \sigma'_{ij} + \delta_{ij}u$, equal to some constant, such that the time derivatives of σ'_{ij} and $\delta_{ij}u$ can be related to each other, e.g. $\dot{\sigma}'_{zz} = -3\dot{u}$. However, the equation $\sigma_{ij} = \text{const.}$ is by no means justifiable for general loading. For a correct theory of three-dimensional consolidation, reference should be made to the papers of Biot (1941, 1955, 1956) and Biot & Willis (1957). Their main improvements are (i) to generalize the definition of effective stress taking into account grain compressibility; and (ii) to admit the case $\sigma_{ij} \neq \text{const.}$, taking into account the time rate of stress. However, the underlying theory of linear elasticity is not realistic as it neglects any dilatancy or contractancy. Other improved consolidation theories (for example, by Gibson *et al.* (1967) or Toorman (1996)) remove a series of the shortcomings of the original equation, but none of the terms introduced so far constitutes a general version capable of dealing with problems with two or three spatial dimensions.

The localized pore-fluid segregation through volcanos can presumably be related to the Mandel–Cryer effect (Cryer 1963; Mandel 1953), which consists of an anomalous pore-pressure increase in consolidation problems with spherical or cylindrical symmetry. However, presently available theoretical and numerical tools do not allow a further analysis of such phenomena. Except for the (rather unrealistic) case of the consolidation of a sphere, the Mandel–Cryer effect has not yet been rigorously analysed due to the lack of appropriate numerical tools.

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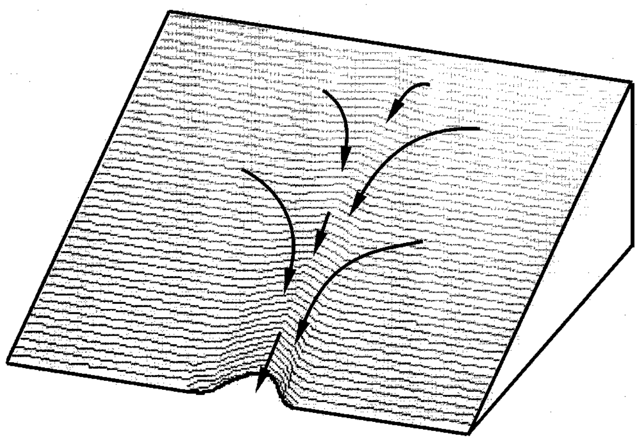

Figure 5. Erosion structure as pattern formation in an initially homogeneous slope.

Understanding the behaviour of liquefied sand will also reveal the key mechanism for any sort of erosion due to flowing water, such as piping, and is closely related to the superficial erosion of natural slopes and embankments: instead of a homogeneous flow downwards, the precipitation water 'prefers' to concentrate and flow in channels (see figure 5).

3. Homogeneous and inhomogeneous consolidation modes

The common basis of the phenomena mentioned in the previous section could be considered in the Darcy–Gersevanov equation (see, for example, Lancellotta 1995), $\boldsymbol{v}^{\mathrm{f}} - \boldsymbol{v}^{\mathrm{s}} = -(k/n)\nabla h$, which relates the gradient of hydraulic head, ∇h, to the *difference* of the velocities of fluid and grains, $\boldsymbol{v}^{\mathrm{f}}$ and $\boldsymbol{v}^{\mathrm{s}}$, respectively. The majority of the analyses in seepage problems refers to the case of fixed grains, i.e. $\boldsymbol{v}^{\mathrm{s}} = \mathbf{0}$, which leads to the special case of Darcy's equation. The important point is that, as soon as the grains are not fixed (as is particularly the case within liquefied soil), then an initial distribution of h does *not* uniquely determine the velocity of the fluid. This non-uniqueness gives rise to the autonomous appearance of vertical drainage pipes and other pattern-formation phenomena.

The researcher who intends to analyse transient problems in granulate–fluid mixtures with two or three spatial dimensions has to take into account a system of equations consisting of the mass- and momentum-balance equations for the granular and fluid phases which incorporate the interaction forces. In so doing, highly nonlinear partial differential equations are obtained.

Let us consider a layer of water-saturated loose sand (see figure 6), which has been completely liquefied due to some dynamic action (e.g. an earthquake). Complete liquefaction means that all effective stresses have been transferred to the pore water. Consequently, the distribution of surplus pore pressure is as shown in figure 6. According to Terzaghi's consolidation theory, the pore-pressure dissipation is related to an homogeneous velocity field: fluid particles move upwards and grains move downwards. What is actually observed is that pore water moves in the radial direction to some vertical channels and then escapes much more quickly and effectively (see figure 7). We thus have a problem with free boundaries because the location

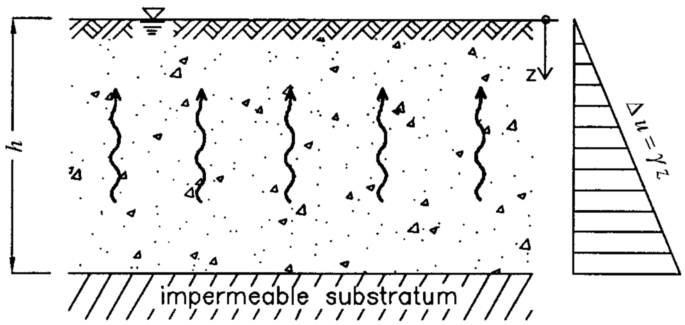

Figure 6. Liquefied sand layer and initial distribution of surplus pore pressure. Homogeneous consolidation mode.

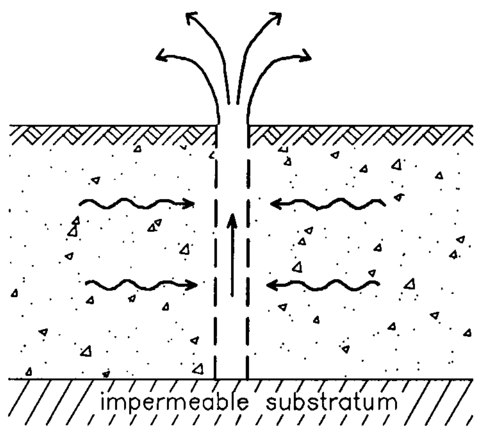

Figure 7. Inhomogeneous consolidation mode: autonomous formation of vertical drains.

and thickness of the vertical channels (or vertical 'drains') is random and *a priori* unknown. The preference of inhomogeneous velocity fields is commonly encountered in the deformation of geomaterials (known as the phenomenon of localization) and is also related to the so-called Boycott effect (Guyon & Troadec 1994).

4. Review of solutions for vertical drains

For the problem of consolidation with vertical drains, many highly sophisticated solutions exist and are compiled by Magnan (1983). The basic approach is by Terzaghi and Rendulic (Magnan 1983), according to which the soil grains move in a vertical direction and the pore water moves in a radial direction. The obtained differential equation for the pore pressure, u, reads

$$c_r\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)=\frac{\partial u}{\partial t}.$$

The solution is obtained by separation of variables and can be represented with Bessel functions, as derived by Abelev (Magnan 1983). More refined theories consider non-constant permeability, i.e. $k = k(e)$, non-constant soil compressibility, creeping soil skeleton, initial gradient, i.e. $v = k(i - i_0)$, and other deviations from the basic assumptions.

Barron and/or Kjellman (Magnan 1983) obtained a mathematically simpler solution by assuming that the settlement of soil is uniform, i.e. it does not depend on r.

In all these solutions, the coupling between the pore pressure, u, and the effective stress, σ'_{ij}, is accomplished by the equation $\sigma_{ij} = \sigma'_{ij} + \delta_{ij}u$ assuming $\sigma_{ij} = \text{const}$. This assumption is, however, not justifiable. In some cases, it is reduced to the assumption $\sigma_{ii} = \text{const.}$, which may, at first glance, appear reasonable for the case of a uniform load on the soil surface. However, the arbitrariness is not removed, but merely transferred to the relation between void ratio, e, and σ'_{ii}, i.e. $\mathrm{d}e = a_v \mathrm{d}\sigma'_{ii}$, ignoring that the compressibility, a_v, of the grain skeleton also depends on the deviatoric parts of $\mathrm{d}\sigma'_{ij}$, as will be seen in the sequel.

A full numerical analysis of the vertical drain problem is not yet available. A series of difficulties must be mastered to present a realistic numerical solution. Besides the difficulty of properly modelling the grain–water interaction (taking into account viscous and, possibly, also inertia coupling), it is realized that the vanishing stiffness at the initial state (which is assumed to be fully liquefied) imposes problems for two reasons (which are virtually interrelated): (i) the constitutive relation for the grain skeleton may break down for vanishing stiffness; and (ii) vanishing stiffness means a bifurcation condition, i.e. non-unique solution. It is due to this bifurcation that the water volcanos can be formed; however, from a numerical point of view, bifurcation imposes problems. It is hoped that these problems can be overcome by using the hypoplastic constitutive equation (Kolymbas *et al.* 1995; von Wolffersdorff 1996), which is known to yield realistic results up to vanishing effective stress level, and of a semi-inverse analysis. A first step towards this analysis is presented by the following approximation.

5. Consolidation with superposition of shear

In the course of our semi-inverse analysis, we shall show that radial drainage is facilitated if a shear motion is superimposed on the deformation of the grain skeleton. We consider a vertical cylindrical pipe in a water-saturated sand layer of thickness H. The pipe (which completely corresponds to a vertical drain) has the radius r_0 and drains a cylindrical region (radius R) of the considered sand layer.

Considering the grains and the pore water as incompressible, the equations expressing mass balance for soil and pore water, respectively, reduce to

$$-\frac{\partial n}{\partial t} + \operatorname{div} \boldsymbol{v}_\mathrm{s} = 0, \tag{5.1}$$

$$\frac{\partial n}{\partial t} + \operatorname{div} \boldsymbol{v}_\mathrm{w} = 0. \tag{5.2}$$

Herein, $\boldsymbol{v}_\mathrm{s}$ and $\boldsymbol{v}_\mathrm{w}$ are the superficial velocities (i.e. discharge rates per unit cross area) of grains and pore water, respectively. From (5.1) and (5.2) follow

$$\operatorname{div} \boldsymbol{v}_\mathrm{s} = -\operatorname{div} \boldsymbol{v}_\mathrm{w}, \tag{5.3}$$

hence

$$\boldsymbol{v}_\mathrm{s} = -\boldsymbol{v}_\mathrm{w} + \operatorname{rot} \boldsymbol{w}, \tag{5.4}$$

where $\boldsymbol{w}$ is an arbitrary vector field. In other words, $\boldsymbol{v}_\mathrm{s}$ and $-\boldsymbol{v}_\mathrm{w}$ differ only by an arbitrary divergence-free vector field $\boldsymbol{x} := \operatorname{rot} \boldsymbol{w}$†, that represents a deviatoric (or pure shear) deformation.

† $\operatorname{rot} \boldsymbol{w}$ is equivalent to $\operatorname{curl} \boldsymbol{w}$.

The point now is that for loose sands, the compressibility is increased if a shear motion is added to the oedometric (i.e. one-dimensional) compression. This is due to the so-called contractancy, i.e. the tendency of loose granular media to reduce their volume when sheared. To demonstrate this effect, we shall consider a set of particular velocity fields. We assume that the $\boldsymbol{v}_\mathrm{w}$ field has a constant divergence, $\mathrm{div}\,\boldsymbol{v}_\mathrm{w} = -a = \mathrm{const.}$, and that the water velocity has only a radial component, i.e. in cylindrical coordinates, r, ϑ, z, we set $\boldsymbol{v}_\mathrm{w} = (v_\mathrm{w}, 0, 0)^\mathrm{T}$. From the boundary condition

$$v_\mathrm{w}(r = R) \stackrel{!}{=} 0,$$

it then follows

$$v_\mathrm{w} = \frac{a}{2r}(r^2 - R^2). \tag{5.5}$$

Now we consider the $\boldsymbol{x}$-field. We choose $\boldsymbol{x} = (v_\mathrm{w}, x_\vartheta, az)$ and require that $\partial x_\vartheta / \partial z = 0$. Furthermore, we require that all functions are independent of ϑ. The dependence of x_ϑ on r can be chosen in such a way that the corresponding component of the stretching tensor $\boldsymbol{D}$†,

$$D_{r\vartheta} = D_{\vartheta r} = \frac{1}{2}\left(\frac{\partial x_\vartheta}{\partial r} - \frac{x_\vartheta}{r}\right), \tag{5.6}$$

is either constant or vanishes asymptotically with $r \to \infty$. The second possibility is certainly more realistic, but the first one was chosen as it is simpler. From

$$\frac{\partial x_\vartheta}{\partial r} - \frac{x_\vartheta}{r} = 2b = \mathrm{const.},$$

we obtain $x_\vartheta = 2br\ln r + r$. This solution is required to be valid for $r_0 < r < R$. Obviously, the field $\boldsymbol{x}$ has zero divergence:

$$\mathrm{div}\,\boldsymbol{x} = \frac{\partial v_\mathrm{w}}{\partial r} + \frac{v_\mathrm{w}}{r} + \frac{\partial x_z}{\partial z} = -a + a = 0.$$

The motion of the grain skeleton is thus obtained in such a way that $\boldsymbol{v}_\mathrm{s} = -\boldsymbol{v}_\mathrm{w} + \boldsymbol{x}$ corresponds to the stretching

$$\boldsymbol{D} = \begin{pmatrix} 0 & b & 0 \\ b & 0 & 0 \\ 0 & 0 & a \end{pmatrix},$$

where an oedometric compression (by virtue of $D_{zz} = a$) is superimposed by a shear motion (by virtue of $D_{r\vartheta} = D_{\vartheta r} = b$). It is now interesting to note that this superimposed shear considerably increases the compressibility of a loose sand. This is to be expected from the behaviour of drained loose sand, which is known to be contractant, and can also be calculated by means of a realistic constitutive law. In figure 8, compression curves are plotted for several values of b/a as obtained from the hypoplastic constitutive law (Kolymbas 1991; von Wolffersdorff 1996) using the above stated stretching, $\boldsymbol{D}$.

The fact that superimposed shear lowers the volumetric stiffness (or, equivalently, increases the compressibility) can possibly explain why inhomogeneous consolidation modes are preferred as compared with homogeneous ones. It should be added, however, that, considering dissipative systems, there are no extremum principles stating that among several possible deformation patterns the one causing minimum work expenses will be selected‡.

† The stretching tensor, $\boldsymbol{D}$, is defined as the symmetric part of the velocity gradient.

‡ The theorem of Helmholtz and Rayleigh of minimum dissipation of creeping flows would here imply

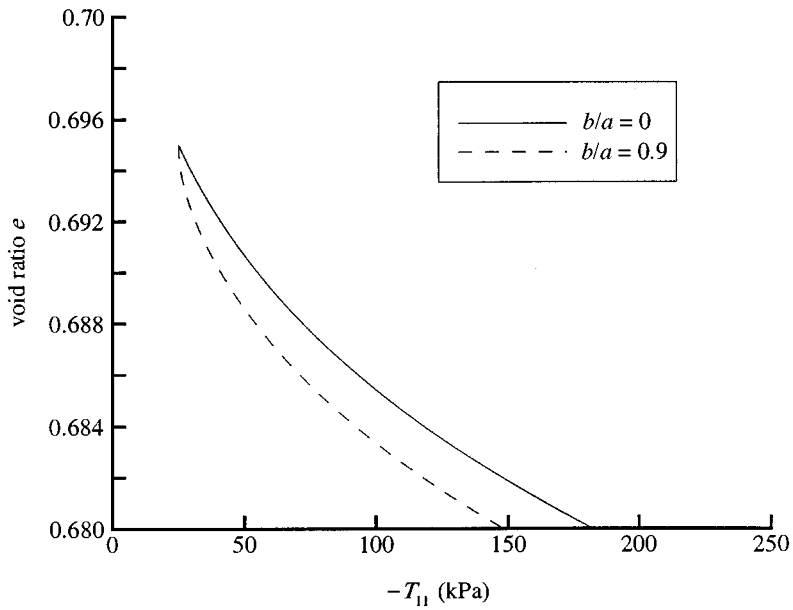

Figure 8. Compression curves for several values of superimposed shear, numerically obtained for loose sand with the hypoplastic constitutive equation. T_{11} is the vertical effective stress.

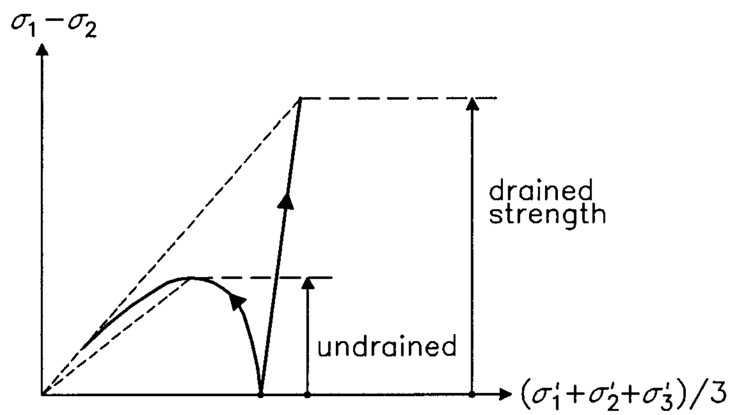

Figure 9. Stress paths at drained and undrained deformation.

6. Liquefaction and soil avalanches

Following Hutter (1996), two limiting cases of avalanches can be distinguished: (i) *flow avalanches*, where the grains contact each other more or less permanently; and (ii) *powder avalanches*, i.e. the turbulent flow of airborne particles. Sturzstroms, debris flows, landslides, rockfalls and most snow avalanches belong mainly to the first type, where fluidization (or liquefaction) is generally accepted to play a decisive role. The transition of slopes to a liquefied state can be explained in soil-mechanics

the minimization of the flow dissipation $\int_V (\gamma_{\rm w}/k)(v^2 - e\boldsymbol{v}_{\rm s}\cdot\boldsymbol{v}_{\rm w})\,{\rm d}V$. The mechanical work rate, $\sigma'_{ij}\dot{\varepsilon}_{ij}$, of the grain skeleton is, however, not considered here.

terms if one considers the undrained behaviour of loose sand. The main principle is outlined in figure 9. The strength (i.e. the maximum stress deviator) is considerably higher at drained than at undrained deformation. Di Prisco *et al*. (1995) analysed several types of instabilities that can be encountered in the course of an undrained loading of water-saturated loose sand. They have shown that such instabilities correspond to negative second-order work (i.e. $\dot{\sigma}_{ij}\dot{\varepsilon}_{ij} < 0$) obtained for particular motions, $\dot{\varepsilon}_{ij}$, that are imposed by internal constraints (incompressibility) and/or symmetry conditions (axisymmetric or plane strain). This implies that water-saturated or submerged slopes become unstable at inclination angles considerably smaller than the friction angle. Darve & Pal (1997) point to the fact that such instabilities can possibly also be encountered even in the absence of pore-water (so-called dry liquefaction).

However, it can hardly be explained how the liquefied state is sustained for durations of several minutes, thus producing the high mobility of large masses of earth material. Several hypotheses have been considered to explain this delayed consolidation. They involve (following Hutter 1996) upward flow of air, hovercraft action, fluidization by high-pressure steam, mechanical fluidization aided by the presence of interstitial dust, lubrication by a thin layer of molten granular material, 'acoustic' fluidization and other assumptions. In this paper, a new explanation is attempted based on the above-stated observation that superimposed shear increases the compressibility of a granular medium. To derive the conclusion we only need the 'classical' consolidation theory, as stated, for example, in the textbook of Taylor (1966), and we may overlook the several modifications provided by more advanced theories. Taking into account the superimposed shear (see figure 8) we assume, for simplicity, that the compressibility of the soil skeleton is very large, say $a_v \to \infty$. The coefficient of consolidation, c_v, vanishes in that case ($c_v = 0$). As known, the time needed for the consolidation of a layer with thickness H amounts to

$$t = \frac{TH^2}{c_v},$$

where the dimensionless time factor T depends on the degree of consolidation (e.g. $T \approx 0.9$ for 90% consolidation). Thus, we see that $t \to \infty$ for $c_v \to 0$, i.e. the consolidation time increases with increasing compressibility of the grain skeleton.

If the contractancy is very pronounced, a_v (and thus c_v) can even become negative. This fact has been addressed by Vardoulakis (1986), who pointed to the possibility of negative diffusion (the so-called backwards heat equation) exhibited by the equation $u_t = cu_{zz}$, for $c < 0$.

7. Liquefaction in the absence of pore fluid

Recent evidence of large landslides on the Moon and on Mars reveals that the pore-water pressure is not necessary for the sustained flow of granular masses†. In the absence of interstitial fluid, the role of pore pressure can be taken by the so-called Reynolds stress. Its nature resides in the fact that transport of momentum can be accomplished not only by permanent contacts between grains but also by the momentum of particles that move freely between two collisions. A very small length

† NASA attributes these landslides to liquefaction caused by trapped gases (for further information, see `http://science.msfc.nasa.gov/newhome/headlines/mgm_images/msad01jan98_2b.htm`). However, this is only an assumption.

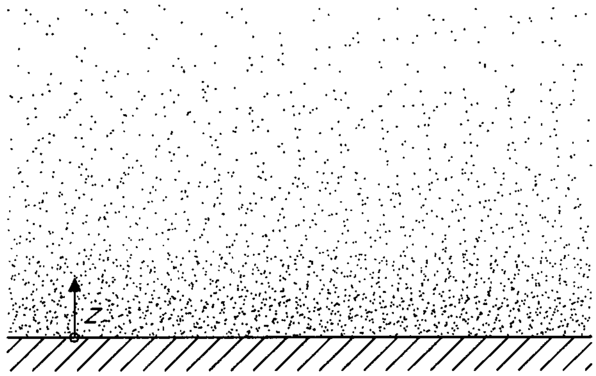

Figure 10. Geometrical set-up for a granular avalanche moving in a horizontal direction.

of free flight is sufficient to sustain velocity fluctuations $\boldsymbol{v}'$, i.e. the velocity of particles $\boldsymbol{v}$ is composed of mean velocity, $\bar{\boldsymbol{v}}$, and of $\boldsymbol{v}'$, i.e.

$$\boldsymbol{v} = \bar{\boldsymbol{v}} + \boldsymbol{v}'.$$

Of course, the void ratio, e, must be larger than e_{max} according to conventional soil mechanics (and, correspondingly, the porosity, n, must be larger than n_{max}). The velocity fluctuation $\boldsymbol{v}'$ gives rise to the Reynolds stress, R_{ij}, which can be derived as

$$R_{ij} = \alpha\rho^0 v_i' v_j' \qquad (\text{or } \boldsymbol{R} = \alpha\rho^0 \boldsymbol{v}' \otimes \boldsymbol{v}'),$$

where ρ^0 is the density of the grains (e.g. $\rho^0 = 2.65\,\mathrm{g\,cm^{-3}}$ for quartz grains) and $\alpha = 1 - n$ is the volumetric fraction of the grains. It is often assumed that the Reynolds stress is a hydrostatic stress, i.e.

$$R_{ij} = \alpha\rho^0 v'^2 \delta_{ij} = r\delta_{ij},$$

with $v' := \sqrt{v_i' v_i'}$. Moreover, it appears reasonable to assume that

$$v' = \beta v,$$

with $v = \sqrt{v_i v_i}$ and $\beta = \text{const}$. This assumption reflects the simple idea that the fluctuations increase with increasing velocity. It may appear non-objective, as $\boldsymbol{v}'$ is an objective quantity, whereas $\boldsymbol{v}$ is not. However, this impression can be removed if we instead write

$$\frac{\mathrm{d}v'}{\mathrm{d}t} = \beta\frac{\mathrm{d}v}{\mathrm{d}t}, \qquad \text{with } v(t=0) = v'(t=0) = 0.$$

The value of β can be expected to be slightly less than unity. Thus, we may set

$$r = \alpha\rho^0\beta^2 v^2.$$

It is worth noting that we have to introduce a fuzzy upper boundary of the sliding granular mass if we want to establish equilibrium in terms of Reynolds stress. To this end, we consider equilibrium in the vertical direction, z, within a granular avalanche, that moves on a nearly horizontal bed (see figure 10).

With $\gamma = \alpha\rho^0 g$ being the bulk specific weight, we obtain from $\mathrm{d}\sigma_z = -\gamma\,\mathrm{d}z$ and $\sigma_z \equiv r$:

$$\int_z^\infty \gamma(z')\,\mathrm{d}z' = r = \alpha\rho^0\beta^2 v^2. \tag{7.1}$$

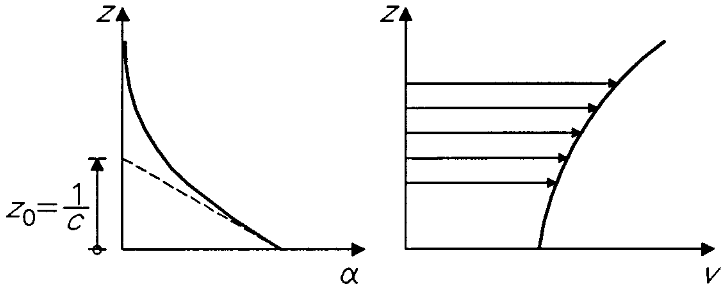

Figure 11. Distributions of the volumetric fraction, α, and velocity, v, over the height, z.

Differentiating with respect to z, and denoting the derivatives by primes, we obtain

$$\alpha g = \alpha' \beta^2 v^2 + \alpha \beta^2 (v^2)'.$$

Setting

$$\frac{g/\beta^2 - (v^2)'}{v^2} = \frac{\alpha'}{\alpha} = -c,$$

with $c = \text{const.} > 0$ we obtain

$$\alpha = \alpha_0 \mathrm{e}^{-cz},$$

i.e. the volumetric fraction of grains decreases exponentially with increasing height z (similar to the density distribution in the atmosphere). This Boltzmann distribution of density of a liquefied granular medium has also been experimentally observed (Warr *et al.* 1995) for the case of a vibration-induced liquefaction. We furthermore obtain

$$v^2 = a\mathrm{e}^{cz} - g/c\beta^2.$$

The integration constant a, can be obtained from the boundary condition, i.e. from equation (7.1) evaluated for $z = 0$. We obtain $a = 2(g/c\beta^2)$, hence

$$v = \sqrt{\frac{g}{c\beta^2}(2\mathrm{e}^{cz} - 1)}.$$

The distributions of α and v against z are plotted in figure 11.

Considering $z_0 = 1/c$ as the height of the avalanche, we obtain the ground velocity $v_0 := v(z = 0)$

$$v_0 = \sqrt{\frac{g z_0}{\beta^2}}. \tag{7.2}$$

Typical values following Hutter (1996) are $v_0 \approx 30$–$60\ \mathrm{m\ s^{-1}}$, $z_0 \approx 1$–5 m, $\rho^0 \alpha \approx 100$–$300\ \mathrm{kg\ m^{-3}}$. Exact measurements will allow the determination of β.

8. Conclusions

The consolidation subsequent to liquefaction can proceed along peculiar patterns that deviate from the traditionally considered homogeneous consolidation that occurs in one spatial direction. This effect can be enhanced by the fact that superimposed shear renders a loose granulate more compressible.

Increased compressibility increases the consolidation time, a fact which can explain the mobility of liquefied earth masses during landslides.

In the absence of interstitial fluid the role of pore pressure can be taken by the Reynolds stress.

The author is indebted to Professor K. Hutter for many valuable remarks.

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